Proposition 2:

A non-trivial chiral vertex operator

$$\Psi(z) : H_{\lambda_0} \otimes H_\lambda \otimes H^*_{\lambda_\infty} \longrightarrow \mathbb{C}$$

exists if and only if the highest weights λ_0, λ and λ_∞ satisfy the quantum Clebsch-Gordan condition at level k.

Conformal invariance (Prop. 4 §5)

$\longrightarrow$ restriction $\Psi_0 : V_{\lambda_0} \otimes V_\lambda \longrightarrow V_{\lambda_\infty}$ is given by $z^{\Delta_\infty - \Delta_{\lambda_0} - \Delta_\lambda} \vec{p}$, $(**)$

where $\vec{p}$ is a basis of

$\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_0} \otimes V_\lambda, V_{\lambda_\infty})$

Ψ is uniquely determined by Ψ_0.

Decompose $\Phi(v, z)$ into $\Phi(v,z) = \sum_{n \in \mathbb{Z}} \Phi_n(v,z)$, such that Φ_n sends $H_\lambda(d)$ to $H_\lambda(d-n)$.

By using Prop. 1 and $(**)$ together with the definition of L_0 one can check the relation

$$[L_0, \Phi_0(v,z)] = \left(z \frac{d}{dz} + \Delta_\lambda\right) \Phi_0(v,z)$$

(exercise)

$\rightarrow$ $\phi(v,z)$, $v \in V_\lambda$ can be written in the form

$$\phi(v,z) = \sum_{n\in\mathbb{Z}} \phi_n(v) z^{-n-\Delta}$$

where $\Delta = -\Delta_{\lambda_\infty} + \Delta_{\lambda_0} + \Delta_\lambda$, $\phi_n(v) : H_{\lambda_0}(d) \to H_{\lambda_\infty}(d-n)$

Proposition 3:

The primary field $\phi(v,z)$, $v \in V_\lambda$, satisfies the relation

$$[L_n, \phi(v,z)] = z^n\left(z\frac{d}{dz} + (n+1)\Delta_\lambda\right)\phi(v,z) \qquad (1)$$

for any integer n.

(exercise)

Interpretation:

$\phi(v,z)(dz)^{\Delta_\lambda}$ is invariant under local hol. conformal transformations, namely we have

$$\phi(v,f(z)) = \left(\frac{df}{dz}\right)^{-\Delta_\lambda} \phi(v,z)$$

$\longrightarrow$ for $f_\varepsilon(z) = z - \varepsilon(z)$ this gives:

$$\delta_\varepsilon \phi(v,z) = \left(\Delta_\lambda \varepsilon'(z) + \varepsilon(z)\frac{d}{dz}\right)\phi(v,z) \qquad (2)$$

In particular, in the case $\varepsilon(z) = \varepsilon z^{n+1}$ the right-hand side of (2) coincides with (1).

-let-hand sides also coincide. Next, define $X(z)$ and $T(z)$ by

$$X(z) = \sum_{n\in\mathbb{Z}} (X\otimes t^n) z^{-n-1},$$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

where $X\in\mathfrak{g}$ and both $X(z)$ and $T(z)$ are formal power series in z. For $u\in H_\lambda$, $\eta\in H_\lambda^*$, $\langle \eta, X(z)u\rangle = \sum_{n\in\mathbb{Z}} \langle \eta, (X\otimes t^n) z^{-n-1} u\rangle$ is expressed as a finite sum, similarly for $T(z)$ ("energy-momentum tensor").

<u>OPE:</u>

$$X(w)\,\Phi(v,z) = \sum_{k\in\mathbb{Z}} w^{-1} z^{-\Delta-k} \sum_{m\in\mathbb{Z}} \left(\frac{z}{w}\right)^m (X\otimes t^m)\,\Phi_{k-m}(v)$$

Assume $|w|>|z|>0$. Using gauge inv., the above expression can be written as

$$\sum_{k\in\mathbb{Z}} w^{-1} z^{-\Delta-k} \sum_{m\geq 0} \left(\frac{z}{w}\right)^m \Phi_k(Xv) + R_1(w-z),$$

where $R_1(w-z)$ is regular in the sense that $\langle \eta, R_1(w-z)\xi\rangle$ is hol. $\forall\, \xi\in H_{\lambda_0},\ \eta\in H_{\lambda_\infty}^*$

$\longrightarrow$ $X(w)\,\phi(v,z)$ in region $|w|>|z|>0$ is analytically continued to $\phi(v,z)\,X(w)$ defined in region $|z|>|w|>0$

"operator product expansion" of $X(w)$ and $\phi(v,z)$

Similarly, we have

$$T(w)\,\phi(v,z)=\left(\frac{\Delta_\lambda}{(w-z)^2}+\frac{1}{w-z}\frac{\partial}{\partial z}\right)\phi(v,z)$$
$$+\,R_2(w-z)$$

in the region $|w|>|z|>0$, where $R_2(w-z)$ is regular in $w-z$. $\longrightarrow$ analytically continued to $\phi(v,z)\,T(w)$ in region $|z|>|w|>0$. In the region $|w|>|z|>0$ we have:

$$T(w)\,T(z)=\frac{c/2}{(w-z)^4}+\frac{2T(z)}{(w-z)^2}+\frac{1}{w-z}\frac{\partial}{\partial z}T(z)$$
$$+\,R_3(w-z),$$

where $R_3(w-z)$ is regular in $w-z$.

$\longrightarrow$ analytically continued to region $|z|>|w|>0$: $T(z)\,T(w)$

Lemma 1:

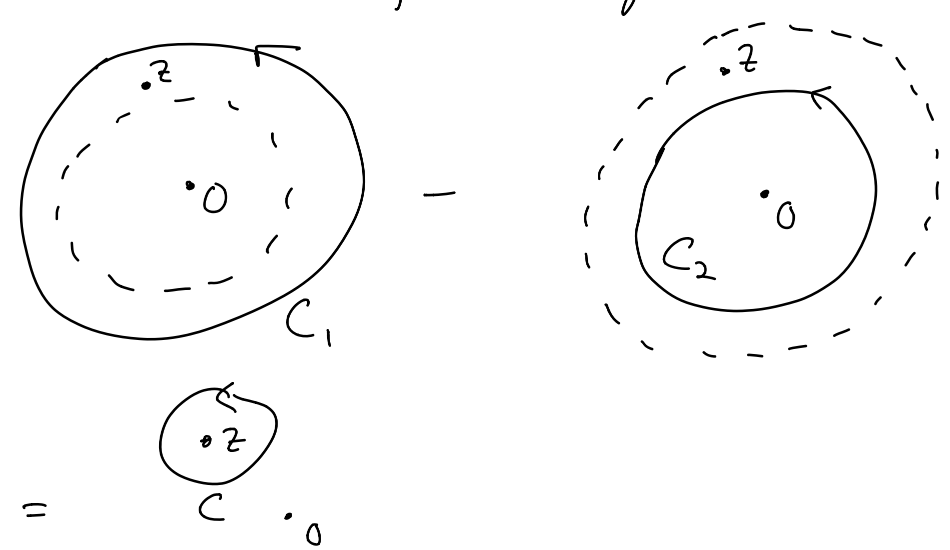
$$[X \otimes t^n, \phi(v,z)] = \frac{1}{2\pi\sqrt{-1}} \int_C \omega^n X(\omega)\, \phi(v,z)\, d\omega,$$

$$[L_n, \phi(v,z)] = \frac{1}{2\pi\sqrt{-1}} \int_C \omega^{n+1} T(\omega)\, \phi(v,z)\, d\omega,$$

where C is an oriented small circle in the ω-plane turning around z counterclockwise.

Proof:

We will show the second equality, the first one is analogous. Fix a point in the ω-plane with coordinate z. Consider the following contours



and the corresponding residues:

$$T(w) = \sum_{n \in \mathbb{Z}} L_n w^{-n-2}$$

$$\longrightarrow L_n \phi(v, z) = \frac{1}{2\pi\sqrt{-1}} \int_{C_1} w^{n+1} T(w) \phi(v, z) \, dw$$

$$\phi(v, z) L_n = \frac{1}{2\pi\sqrt{-1}} \int_{C_2} w^{n+1} \phi(v, z) T(w) \, dw$$

$$\Longrightarrow \int_{C_1} w^{n+1} T(w) \phi(v, z) \, dw - \int_{C_2} w^{n+1} \phi(v, z) T(w) \, dw$$

$$= \int_{C} w^{n+1} T(w) \phi(v, z) \, dw$$

$\square$

Combining Lemma 1 and OPE (3), we obtain commutator (1).

Next, we explain how (4) gives rise to Virasoro Lie algebra. Let γ be a circle in the z-plane with parameter $z = re^{2\pi\sqrt{-1}\,\theta}$, $0 \le \theta \le 1$.

Take circles C_1 and C_2 in the w-plane and suppose $r_2 < r < r_1$. Then we have

$$L_m L_n = \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int_{\gamma} \int_{C_1} w^{m+1} z^{n+1} T(w) T(z) \, dw \, dz$$

$$L_n L_m = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\gamma \int_{C_2} w^{m+1} z^{n+1} T(z) T(w)\, dw\, dz$$

Thus for a circle C as in the above picture, we get

$$[L_m, L_n] = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\gamma \left(\int_C w^{m+1} z^{n+1} T(w) T(z)\, dw \right) dz$$

Combining with OPE (4) we obtain the Virasoro Lie algebra. $\square$

<u>Definition:</u>

For a transformation f of the complex plane we introduce the "Schwarzian der."

$$S(f,z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

<u>Lemma 2:</u>

For a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C},\ ad-bc=1,$$

the equality $S(f,z)=0$ holds for any $z \in \mathbb{C}$. Conversely, if $S(f,z)=0$ for any $z \in \mathbb{C}$, then f is a Möbius transformation

From the OPE of the energy momentum tensor we get

$$(*) \quad \delta_\varepsilon T(z) = \varepsilon(z)\frac{\partial}{\partial z} T(z) + 2\varepsilon'(z) T(z) + \frac{c}{12}\varepsilon'''(z)$$

$\longrightarrow$ $T(z)$ is <u>not</u> a co-variant tensor of order 2

For $\varepsilon = z^{n+1}$, $n=-1,0,1$, f_ε generates a global Möbius trf. and $\frac{c}{12}\varepsilon''' = 0$.

Integral form of $(*)$:

<u>Proposition 4:</u>

For a hol. transformation $w = f(z)$, we have

$$T(z) = \left(\frac{\partial w}{\partial z}\right)^2 T(w) + \frac{c}{12} S(f, z)$$